REPORT ON OUR PROJECT
"*PRESCRIBED SCALAR CURVATURE ON OPEN MANIFOLDS*"
AT THE HAUSDORFF INSTITUTE (SEPTEMBER – DECEMBER 2011)

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The project was centered on the scalar curvature of Riemannian metrics on open, i.e. noncompact connected, manifolds. In the first part of the project we examined properties of a conformal invariant, the Yamabe constant; this subproject led to a publication and is now finished. In the second part we investigated — and still investigate — to which extent a given function on an open manifold can be realised as the scalar curvature of a Riemannian metric.

1. THE YAMABE CONSTANT

The *Yamabe constant* $Y(g) \in [-\infty, \infty[$ of a nonempty Riemannian manifold (M, g) of dimension $n \geq 3$ is the infimum of $Q_g(v)$ over all compactly supported smooth functions $v\colon M \to \mathbb{R}_{\geq 0}$ which are not identically 0; here $Q_g(v) \in \mathbb{R}$ is defined by (see [16])

$$Q_g(v) = \frac{\int_M \big(\frac{4(n-1)}{n-2}|\mathrm{d}v|_g^2 + \mathrm{scal}_g v^2\big)\mathrm{d}\mu_g}{\big(\int_M v^{2n/(n-2)}\mathrm{d}\mu_g\big)^{(n-2)/n}} \quad .$$

$Y(g)$ depends only on the conformal class of g. The existence of constant scalar curvature metrics in this conformal class is closely related to $Y(g)$. Therefore $Y(g)$ is a very important and especially on closed manifolds extensively investigated quantity [1],[2],[14],[16],[18]. The noncompact case has been considered for instance in [13, 19, 6].

Bérard Bergery proved in [3] that on a *compact* manifold M, the function $g \mapsto Y(g)$ is continuous with respect to the C^2-topology on the space of Riemannian metrics on M. We generalized this to noncompact manifolds in the following way:

1.1. **Theorem.** [7] *For every nonempty manifold M of dimension ≥ 3, the map $g \mapsto Y(g) \in [-\infty, \infty[$ is continuous with respect to the fine (a.k.a.* strong *or* Whitney*) C^2-topology on the space of Riemannian metrics on M.*

The definition of the fine C^k-topology can be found in [10], for instance. On compact manifolds, it is equal to the compact-open C^k-topology; but in the noncompact case, it is much finer (not even first countable, in particular not metrizable). We also discussed to which extent the Yamabe constant is continuous with respect to coarser topologies on the space of Riemannian metrics. On each open manifold, continuity with respect to any compact-open C^k-topology fails at every metric. Continuity with respect to any uniform C^k-topology fails at least at certain metrics on some manifolds.

This part of the project resulted in the publication [7], which contains also other results that we are not going to discuss in the present report.

2. PRESCRIBED SCALAR CURVATURE

Kazdan–Warner proved in 1975 [12, Theorem 1.4] that on every open manifold M of dimension $n \geq 2$ *which can be embeddded into a compact n-manifold*, every smooth function $s\colon M \to \mathbb{R}$ is the scalar curvature of a smooth Riemannian metric. Every manifold (without embeddability assumption) of dimension ≥ 3 admits a complete Riemannian metric of constant negative scalar curvature [4]. (In contrast, many open manifolds do not admit a *complete* Riemannian metric of positive scalar curvature [5], and many closed manifolds do not admit a scal > 0 metric either [17].)

However, it is still not known [11] whether our following conjecture is true:

Conjecture. *On every open manifold M of dimension ≥ 2 (without further topological assumptions), every smooth function $s\colon M \to \mathbb{R}$ is the scalar curvature of a smooth Riemannian metric.*

We intend to solve this problem in our second project, as a corollary to much sharper results. Namely, we consider the following problem: Given are a smooth function $s\colon M \to \mathbb{R}$, a codimension-zero submanifold-with-boundary A of M which is a closed subset of M, and a smooth metric g on M whose scalar curvature is equal to s on the set A. The task is to decide whether there exists a metric g' on M which coincides with g on A and whose scalar curvature is s on all of M.

We splitted this project into three steps. First we asked whether one can always find a metric g_1 that coincides with g on A and whose scalar curvature is larger than s on $M \setminus A$. This part is finished, the answer is always *yes* if each connected component of $M \setminus A$ is not relatively compact in M. The corresponding article is in preparation [8].

Using this information, we show in the second step that for all $\varepsilon \in C^\infty(M, \mathbb{R}_{>0})$ there is a metric g_2 that coincides with g on A, fulfills $s \leq \mathrm{scal}_{g_2} \leq s + \varepsilon$ on $M \setminus A$, has C^0-distance $\leq \varepsilon$ from g, and whose C^1-distance from g can be controlled universally in a suitable sense. This C^1-control improves a result of Lohkamp [15]. This second step of the project is finished as well, the article is in preparation [9].

In the third step, we want to apply the other steps in order to solve the problem mentioned above, in particular to prove the conjecture. This is still (March 2013) work in progress.

3. CONCLUDING REMARKS

We greatly enjoyed the quiet atmosphere at the Hausdorff Institute, which helped us to focus on the progress of our project. We liked the distraction caused by the coffee breaks as well, though. The cake was excellent.

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FINAL REPORT

In the period of September to December, 2011, our group, composed by the following young researchers:

Fei Han (National University of Singapore)
Alexander Kahle (Universität Göttingen)
Arturo Prat-Waldron (Max Planck Institute for Mathematics)
David Corbett Redden (Michigan State University)
Konrad Waldorf (Universität Regensburg)

was hosted by the Hausdorff Research Institute for Mathematics (HIM) in the framework of Junior Trimester Program "Differential Geometry".

HIM provided us a great environment for research. Working together in HIM, the group members can share ideas and discuss specific problems intensively. We had a Field Theory seminar, usually once a week, where new results or ideas were reported. Working in HIM also provided us opportunity to learn from experts. The visiting of Prof. Dr. Teichner to our group benefited us a lot. HIM provided financial assistance to nonlocal visitors to our group too. These visitors included Michael Joachim (Universität Münster), Thomas Nikolaus (Universität Regensburg), Dmitry Pavlov (Universität Münster) and Urs Schreiber (Universiteit Utrecht).

Our group specialized in the field theory aspects of differential geometry. Cohomology and K-theory are classical theories in differential geometry and algebraic topology. The celebrated Atiyah-Singer index theorem equals the index of the Dirac operator on a spin manifold to the $\widehat{A}$-genus, a topological invariant of the manifold. In the family case, i.e. for a family of spin manifolds and the corresponding family of Dirac operators, K-theory of the parametrizing space is the home of the family index. The Witten genus is a q-deformation of the $\widehat{A}$-genus, which is an integral modular form and can be formally thought of as the S^1-equivariant index of the Dirac operator on the free loop space of a string manifold. The theory of topological modular forms (TMF) developed by Hopkins and Miller is the home for a family of Witten genus, just as the K-theory is the home for a family of the $\widehat{A}$-genus. Unlike cohomology and K-theory, which can be described by geometric cocycles in various manners , TMF is so far only defined in the realm of homotopy theory. The Stolz-Teichner program aims to geometrize TMF by using 2 dimensional supersymmetric field theories as cocycles.

0 and 1 dimensional supersymmetric field theories have already been shown to give cohomology and K theory by Stolz, Teichner and their coauthors. Our group focused on topics related to the Stolz-Teichner program.

Differential cohomology is the refinement of the ordinary cohomology. Redden studied differential cocycles and wrote a paper about this. ("Trivializations of differential cocycles", [arXiv:1201.2919]).

As cohomology, K-theory also has smooth extension, the differential K-theory. There are several existing models for differential K-theory. As mentioned above, in the Stolz-Teichner program, K-theory has been related to 1 dimensional supersymmetric field theories. Kahle and Prat-Waldron were working on the joint project "1|1-Euclidian Field Theories and Differential K-theory" aiming to give a new model for differential K-theory with 1|1-Euclidian Field Theories over manifolds being cocycles.

In the equivariant aspects of the theory, Han jointed in a project of Prof. Dr. Teichner and Prof. Dr. Stolz, which studied the supersymmetric field theory realization of the Cartan model of equivariant cohomology as well as realization of the Bismut-Chern character as a dimension reduction functor. The relevant paper is in progress.

The Dirac operator on free loop space is not yet defined rigorously. To understand the geometry of free loop space is big challenge. The transgression map from manifold to its free loop space relates geometric objects on the original manifold to geometric objects on the free loop space. Waldorf and our visitor Nikolaus investigated along this direction, which was represented in the paper: "Lifting Problems and Transgression for Non-Abelian Gerbes", [arxiv:1112.4702].

We are indebted to HIM for the supportive working environment and the financial assistance. Our stay at HIM has benefited our research and is likely to have a lasting impact on our future research.

ACTIVITY REPORT

ROBERT HASLHOFER AND HANS-JOACHIM HEIN

Main project: All currently known compact Ricci-flat manifolds have special holonomy, i.e. they are either Calabi-Yau, hyper-Kähler, G_2, Spin(7), or flat. It is an outstanding question whether this is necessarily the case—more precisely, whether a compact, simply-connected, irreducible Ricci-flat n-manifold can have generic holonomy $\mathrm{SO}(n)$. There seems to be little hope for a general existence theorem for the Riemannian Einstein equations that would allow one to answer such questions. On the other hand, it is sometimes possible to find compact manifolds that are *approximately* Ricci-flat, by gluing together truncations of various singular or noncompact Ricci-flat building blocks (which are typically easier to construct), such that in addition the holonomy group of these approximators is far from being special. The *singular perturbation method* in geometric analysis then suggests to apply the implicit function theorem to construct Ricci-flat holonomy $\mathrm{SO}(n)$ metrics nearby.

Singular perturbation has been applied with great success to several elliptic geometric problems, notably the construction of self-dual Yang-Mills connections over compact 4-manifolds (Taubes) and of complete surfaces of constant mean curvature in $\mathbb{R}^3$ (Kapouleas). Recent work of Biquard seems to provide the first instance of a successful application of such ideas to the full Einstein equations, albeit in a noncompact setting which makes it easier to push obstructions off to infinity.

During our time at HIM we carefully studied the work of Taubes, Kapouleas and Biquard, aiming to acquaint ourselves with their techniques. However we came to realize in the end that there exist formidable obstructions to carrying out such a program for compact Ricci-flat manifolds.

Other projects:

RH completed a project with Stuart Hall (Buckingham) and Michael Siepmann (ETH Zürich) on the stability inequality for Ricci-flat cones [4]. The excellent working conditions at HIM as well as conversations with other trimester participants have been particularly helpful for this.

RH, HH and Aaron Naber (MIT) had some very fruitful discussions on quantitative stratification for geometric equations. This initiated two projects [1, 2] by RH joint with Aaron and Jeff Cheeger (Courant) on quantitative stratification for the mean curvature and harmonic map flows. Earlier in 2011 in Oberwolfach HH and Aaron had begun to study a related question (local regularity) for the Ricci flow; work on this project continued at HIM and has since resulted in the paper [7].

HH also continued a collaboration on asymptotically conical Calabi-Yau manifolds [3] with Ronan Conlon (McMaster), who was visiting the Max-Planck Institute in Bonn at the time. In particular, a talk by Frank Reidegeld (Dortmund) in the HIM seminar sparked an idea for constructing a new class of Calabi-Yau manifolds from pairs of flag varieties [3, Part I, Section 4].

Moreover, HH made some progress on a long-term project of his to do with the classification of gravitational instantons. This benefited greatly from conversations with HIM trimester participant Johannes Nordström (Imperial College London), and with Olivier Biquard and Philip Boalch (ENS Paris) who were both visiting HIM for the hyper-Kähler workshop in December.

Lectures:

In the trimester seminar, RH gave a survey talk on compactness theorems in geometry and a more specialized talk on singularities in 4d Ricci flow.

HH gave a survey talk on the geometry of compact Einstein manifolds in the trimester seminar and lectured on gravitational instantons in the hyper-Kähler workshop.

Date: March 19, 2013.

Activities organized:

RH organized, together with Panagiotis Konstantis (Tübingen) and Jan Swoboda (MPI Bonn), a workshop on geometric flows. We had 18 talks in total, including 6 talks by participants of the HIM trimester. The interaction between participants working on a variety of related topics was highly fruitful, and we very much appreciate the excellent support by the HIM.

Visitors:

RH invited Reto Müller (Imperial College London) to stay at HIM a couple of extra days around the workshop on geometric flows. We had some initial discussions on Łojasiewicz-Simon inequalities for the Ricci flow, which eventually resulted in the paper [6].

HH invited Mark Haskins (Imperial College London) to visit HIM for two weeks in December. In the course of this visit we developed an idea for a joint paper with Johannes Nordström, which has since appeared [5]. We also worked on the planning of a 2-week conference on Ricci curvature and Kähler-Einstein metrics to take place at ICMS Edinburgh this year. In addition Mark collaborated with Johannes and Diarmuid Crowley (MPI) on the differential topology of G_2-manifolds.

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Variational Methods and Special Holonomy

Jan Swoboda*, Hartmut Weiß† and Frederik Witt‡

1 On the project

Metrics of special holonomy are at the crossroads of Riemannian, symplectic and complex geometry and play an important rôle in modern developments of differential geometry and theoretical physics. Further advances in the understanding of these metrics are clearly desirable. Our research project approached these metrics from a variational point of view which more generally leads to the investigation of various deformation and moduli spaces of special geometric structures. Within this framework we worked on several subprojects.

1.1 Energy functionals and soliton equations for G_2-forms (H. Weiß and F. Witt)

We extended short-time existence and stability of the Dirichlet energy flow as introduced in an earlier paper of ours [6] to a broader class of energy functionals. Furthermore, we derived some monotonely decreasing quantities for the Dirichlet energy flow and investigated an equation of soliton type. In particular, we showed that nearly parallel G_2-structures satisfy this soliton equation and studied their infinitesimal soliton deformations. This paper was finalised at HIM.

1.2 A spinorial energy functional: critical points and gradient flow (B. Ammann, H. Weiß and F. Witt)

On the universal bundle of unit spinors we studied a natural energy functional whose critical points, if $\dim M \geq 3$, are precisely the pairs (g, ϕ) consisting of a Ricci-flat Riemannian metric g together with a parallel g-spinor ϕ. In particular, the underlying metric is of special holonomy. We investigated the basic properties of this functional and studied its negative gradient flow. In particular, we proved short-time existence and uniqueness for this flow. Furthermore, we investigated the moduli space of critical points and showed smoothness under additional assumptions. This paper was initiated at HIM and written jointly with Bernd Ammann (Universität Regensburg).

1.3 Morse homology for the Yang–Mills gradient flow (J. Swoboda)

We used the Yang-Mills gradient flow on the space of connections over a closed Riemann surface to construct a Morse chain complex. The chain groups are generated by Yang-Mills

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connections. The boundary operator is defined by counting the elements of appropriately defined moduli spaces of Yang-Mills gradient flow lines that converge asymptotically to Yang-Mills connections. This paper was finalised at HIM.

1.4 A symplectically non-squeezable small set and the regular coisotropic capacity (J. Swoboda und F. Ziltener)

We proved that for $n \geq 2$ there exists a compact subset X of the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{2}$, such that X has Hausdorff dimension n and does not symplectically embed into the standard open symplectic cylinder. The second main result gives a lower bound on the d-th regular coisotropic capacity, which is sharp up to a factor of 3. For an open subset of a geometrically bounded, aspherical symplectic manifold, this capacity is a lower bound on its displacement energy. The proofs of the results involved a certain Lagrangian submanifold of linear space, which was considered by M. Audin [1] and L. Polterovich [5]. Essential parts of this paper were written together with Fabian Ziltener (KIAS Seoul) at HIM.

1.5 SQuaRE project “Nonlinear analysis and special geometric structures” (R. Mazzeo, J. Swoboda, H. Weiß and F. Witt)

Apart from taking individual projects further we quickly focused on a new project which involved all three of us. Inspired by very useful conversations with Justin Sawon on the moduli space of Higgs bundles and related issues as well as by our jointly organised conference on hyperkähler geometry, we decided to attack Hausel's conjecture [2, 3] on the L^2-cohomology of Hitchin's Higgs bundle moduli space [4]. Together with Rafe Mazzeo (Stanford University) we formulated a SQuaRE project “Nonlinear analysis and special geometric structures" (http://www.aimath.org/research/squares.html) which was accepted in December 2011.

2 Our stay at HIM

During our stay at HIM we interacted quite a lot with other participants such as H.-J. Hein, R. Haslhofer, R. Glover, J. Sawon and J. Nordström through various topics such as geometric flows, generalised geometry (à la Hitchin), hyperkähler geometry and G_2-manifolds. As our guests we were pleased to have with us Fabian Ziltener (KIAS Seoul) and Michael Struwe (ETH Zürich) who also delivered a small lecture series on *Conformal metrics of prescribed Gauss curvature on closed surfaces of higher genus*. Apart from giving talks at the MPI Seminar, the workshop on geometric flows and in the trimester seminar (co-)organised by J. Swoboda, we organised (jointly with J. Sawon) the workshop on hyperkähler geometry. As a whole we found very good working conditions and our stay at HIM very stimulating.

3 Publications

- B. Ammann, H. Weiß and F. Witt, *A spinorial energy functional: critical points and gradient flow*, submitted.
- J. Swoboda, *Morse homology for the Yang-Mills gradient flow*, J. Math. Pures Appl. **98**, 160–210, 2012.
- J. Swoboda and F. Ziltener, *A symplectically non-squeezable small set and the regular coisotropic capacity*, to appear in: J. Symplectic Geometry

- H. Weiß and F. Witt, *Energy functionals and soliton equations for* G_2*-forms.* Ann. Global Anal. Geom. **42** (2012) no. 4, 585–610.

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